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On Boundedness of Magnetohydrodynamic Flows. II

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I. INTRODUCTION

The subject of plasma physics and magnetohydrodynamics was initiated a relatively short time ago by astrophysicists and has since assumed considerable importance in engineering. Theoretical hydrodynamics, however, has long attracted the attention of scientists working in a variety of specialized fields. In this connection we mention a paper by E. Hopf (1941) in which he considered the boundedness of hydrodynamic flow in an infinitely long cylindrical pipe [1]. He made this problem of infinite type, as he calls it, artificially a problem of finite type by imposing the condition of spatial periodicity on the velocity vector u. The assumption of spatial periodicity is, however, physically unrealistic and the recent theoretical work has dealt with the development from a much more general and realistic point of view [2, 3]. More general problems which arise from the addition of electromagnetic effects have now been considered by the author in his recent study of the stability theory of flows [4, 5]. This paper is intended to study one such problem, namely, the boundedness of hydromagnetic flows in an infinitely long cylindrical pipe of an arbitrary cross-section with an applied radial magnetic field.

II. NATURE OF THE PROBLEM

The major portion of theoretical literature in magnetohydrodynamics is based on the simplifying assumption of uniform electrical conductivity throughout the flow field [6, 7]. It is a well-established fact that for high temperature gases, as working fluids, the conductivity is a strongly dependent function of the temperature [8, 9]. In operational devices, due to cooling of the region, e.g., a channel, a region of low conductivity exists in the cool thermal boundary layer near the wall. The dissipative Joule heating, which cannot be neglected in magnetohydrodynamic analysis [10] has an additional effect on the nonuniform temperature distribution. Hence, the assumption of uniform electrical conductivity may be considered as questionable and the effects of nonuniformity on the flow must be taken into consideration. We carry this generality of nonuniform conductivity right through the formal solution of our main result except in an auxiliary section on magnetic energy where it is restricted to be uniform in space.

We are given an infinitely long cylindrical pipe Ω which is bounded by a single and sufficiently smooth nonconducting surface Σ. The fluid moves inside the fixed material pipe and the physical boundary condition of the flow problem is that at each instant the fluid adheres to the pipe. Mathematically speaking, this means that the boundary of the pipe does not change with time and the velocity vector u vanishes on the boundary. The conductivity of the fluid is taken to be finite, positive, and nonuniform except in an auxiliary section on magnetic energy where it is restricted to be uniform in space. It is assumed to be and to remain isotropic inspite of presence of the magnetic field. The applied electric field is assumed to be zero.

In what follows we establish the existence of eventual bounds (i.e., as $t \to \infty$) for the kinetic energy and the dissipation of kinetic energy along the regular solutions of magnetohydrodynamic equations for large values of the Reynolds number. In an earlier paper [5] we established the stability of laminar flow in an infinitely long cylindrical pipe for sufficiently large values of the viscosity parameter μ. In majority of the hydromagnetical arrangements (boundary conditions) the laminar flow is unstable and other forms of solutions (turbulence) are observed. An exception to this rule is the case $u = 0$ on Σ where the flow tends to the laminar solution (the state of rest: $u = 0$ in this case) for any value of $\mu > 0$.

We use the notation $x = (x_1, x_2, x_3)$ for a point in the flow space and $u = (u_1, u_2, u_3)$ denotes the flow velocity vector. $B = (B_1, B_2, B_3)$ is the applied radial magnetic field. We consider four quadratic functionals:

$$K_\alpha = \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} u_i u_i \, dx \tag{1}$$

$$W_\alpha = \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} B_i B_i \, dx \tag{2}$$

$$I = \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} u_{i,j} u_{i,j} \, dx \tag{3}$$

$$U = \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \frac{J_i J_i}{\sigma} \, dx, \tag{4}$$

where J_i are the components of the current density and $dx = dx_1 \, dx_2 \, dx_3$. K_α and μI_α are the average kinetic energy and the average dissipation of kinetic energy of the fluid contained in a section Ω_1: $-\alpha \leqslant x_1 \leqslant \alpha$ of the pipe. W_α and U_α are the average magnetic energy and the average dissipation of magnetic energy into Joule heat at the rate of J^2/σ per unit volume of the magnetic field occupying the region Ω_1.

III. Fundamental Equations

1. *The Equations and Their Validity*

We consider a viscous, homogeneous, incompressible and electrically conducting fluid of finite conductivity. The equations governing the motion of such a fluid in an infinitely long cylindrical pipe with an applied radial magnetic field may be taken as

$$\frac{\partial u_i}{\partial t} + u_{i,j}u_j = -p_{,i} - \sigma B_j B_j u_i + \mu u_{i,jj}, \tag{5}$$

$$\frac{\partial B_i}{\partial t} = \epsilon_{ijk}\epsilon_{klm}(u_l B_m)_{,j} + \eta B_{i,jj}, \tag{6}$$

$$u_{i,i} = 0. \tag{7}$$

Here, and henceforth, we set the density of the fluid equal to one and assume that the coefficient of viscosity μ is constant. σ is the electrical conductivity and $\eta(\sim 1/\sigma)$ is the coefficient of magnetic diffusivity. The auxiliary quantity p in the momentum transport equation is the pressure. The space of magnetohydrodynamics is a Euclidean space of three dimensions and the coordinate system employed is cartesian rectangular. The units used are rationalized Gaussian with $c = 1$.

2. *Basic Assumptions and Auxiliary Results*

In order to determine a unique solution of our flow problem, we make the following mathematical assumptions:

(i) *Local side conditions*

u_i, $u_{i,j}$, $u_{i,jj}$, $\dfrac{\partial u_i}{\partial t}$ (and consequently p, $p_{,i}$ and B_i^2), $B_{i,j}$, $B_{i,jj}$, and $\dfrac{\partial B_i}{\partial t}$ are continuous in $\Omega + \Sigma$ (the interior and on the boundary of the pipe), $\quad t \geqslant 0,$ (8)

and

$$u_{i,i} = 0 \quad \text{in} \quad \Omega \tag{9}$$

$$u_i = 0 \quad \text{on} \quad \Sigma. \tag{10}$$

$$\int_{x_1=\text{const}} u_1 \, dS = N \tag{11}$$

where N is the given crossflux independent of time.

(ii) *Side conditions at infinity*

$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} u_1(x, t)\, h(x)\, dx$ exists for each fixed finite t and for each function $h(x_2, x_3)$ continuous in the interior and on the boundary of the cross-section. (12)

$\lim_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} B_j B_j u_i u_i\, dx$ exists for each fixed finite t. (13)

$\lim_{\alpha\to\infty} \frac{1}{4\alpha} \int B_i B_i B_j B_j\, dx = \lim \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} B^4\, dx$ exists for each fixed finite t. (14)

$K = \lim_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} u_i u_i\, dx$ exists for each fixed finite t. (15)

$I = \lim_{\alpha\to\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} u_{i,j} u_{i,j}\, dx$ exists for each fixed finite t. (16)

$U = \lim_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} J_i J_i\, dx$ exists for each fixed finite t. (17)

Clearly the existence of (14) implies the existence of $\text{limit}_{\alpha\to\infty} W_\alpha$. To see this we merely have to apply the Schwarz inequality to the right-hand side of (2). Hence,

$W = \lim_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} B_i B_i\, dx$ exists for each fixed finite t. (18)

(iii) *Boundedness conditions*

For each fixed finite t, there exist positive numbers α_0, C_i, $i = 1, 2, 3, 4, 5$ such that for $\alpha > \alpha_0$,

$$K_\alpha \leqslant C_1, \tag{19}$$

$$I_\alpha \leqslant C_2, \tag{20}$$

$$U_\alpha \leqslant C_3, \tag{21}$$

$$\frac{1}{\alpha} \int_{-\alpha}^{\alpha} B_i B_i B_j B_j\, dx \leqslant C_4, \tag{22}$$

$$\frac{1}{\alpha} \int_{-\alpha}^{\alpha} (p - \bar{p})^2\, dx \leqslant C_5, \tag{23}$$

where $\bar{p}$ is the mean pressure on the cross-section $x_1 = \text{const}$. It is again easily verified from (22) and (2) that W_α is bounded. Hence, there exists a positive constant C_6 such that for $\alpha > \alpha_0$,

$$W_\alpha \leqslant C_6 . \tag{24}$$

Consider an arbitrary vector field a such that

$$a = (a_1, 0, 0); \qquad a = a(x_2, x_3) \qquad \text{and} \qquad a_i = u_i \qquad \text{on} \qquad \Sigma. \tag{25}$$

We suppose that this vector field a satisfies all the smoothness properties as postulated for u in (8). An immediate consequence of (25) is that $a_{i,i} = 0$ is automatically satisfied. In fact, the rather stringent requirement concerning smoothness as stated in (8) may be relaxed considerably. The main point is that the use of Gauss-Green theorem is permissible in what follows. It is easily verified that with this vector field a,

$$K(u - a) = \operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} (u_i - a_i)(u_i - a_i)\, dx, \tag{26}$$

$$I(u - a) = \operatorname*{limit}_{\alpha\to\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} (u_i - a_i)_{,j} (u_i - a_i)_{,j}\, dx, \tag{27}$$

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha} \int_{-\alpha}^{\alpha} B_j B_j (u_i - a_i)(u_i - a_i)\, dx \text{ exist for each fixed finite } t \geqslant 0 \tag{28}$$

and

$$K_\alpha(u - a) \leqslant C_1^* \tag{29}$$

$$I_\alpha(u - a) \leqslant C_2^* \tag{30}$$

for each fixed finite $t \geqslant 0$ and $\alpha > \alpha_0$; C_1^* and C_2^* being positive numbers.

In writing down the equations of motion we have assumed that all flow velocities are small in comparison to the velocity of light so that the relativistic effects such as the displacement currents may be neglected. The requirement $B_{i,i} = 0$ has been imposed as the initial condition [11]. We have assumed that the applied electric field is zero. There may, however, be a small amount of separated charge at the boundary due to the motion of conducting fluid across the field lines. Such effects as the charge accumulation and the boundary effects on the wall will be assumed negligible.

We will use an arbitrary function $f(x_1) = f(x_1, \alpha, \Delta\alpha)$ having the following properties:

$$f(x_1) = f(-x_1), \qquad 0 \leqslant f \leqslant 1 \tag{31}$$

$$f(x_1) = \begin{cases} 1, & 0 \leqslant x_1 \leqslant \alpha - \Delta\alpha \\ \epsilon C'', & \alpha - \Delta\alpha \leqslant x_1 \leqslant \alpha \\ 0, & x_1 \geqslant \alpha \end{cases} \tag{32}$$

$$|f_{,1}(x_1)| = \begin{cases} 0, & x_1 = \pm\,\alpha \\ < \dfrac{2}{\Delta\alpha}, & x_1 \geqslant \alpha \end{cases} \tag{33}$$

$$|f_{,11}(x_1)| < \frac{m}{(\Delta\alpha)^2}, \qquad \alpha - \Delta\alpha \leqslant x_1 \leqslant \alpha, \tag{34}$$

where $\Delta\alpha \geqslant 1$ and $\text{limit}_{\alpha\to\infty}(\Delta\alpha/\alpha) = 0$. The choice of this $\Delta\alpha$ will be made later; m being a positive constant.

The simplest way to realize such an f is to set it up in the following form:

$$f(x_1) = \varphi\left(\frac{\alpha - x_1}{\Delta\alpha}\right); \qquad \varphi \in C'' \qquad \text{in} \qquad 0 \leqslant \frac{\alpha - x_1}{\Delta\alpha} \leqslant 1. \tag{35}$$

φ has the following additional properties:

$$\varphi(0) = \varphi_{,1}(0) = 0, \qquad \varphi(1) = 1, \qquad \varphi_{,1}(1) = 0$$

$$|\varphi_{,1}| < 2 \qquad \text{and} \qquad \varphi_{,11}(|\varphi_{,1}|)^{-1/2} < M\ (\text{finite}) \tag{36}$$

Lastly we introduce another function ψ such that

$$\psi = |f_{,1}|^{1/2} = (\varphi_{,1})^{1/2}(\Delta\alpha)^{-1/2}, \tag{37}$$

$$\psi' = \{(\varphi_{,1})^{1/2}\}_{,1}(\Delta\alpha)^{-3/2}. \tag{38}$$

We now state the main result of this paper in the form of a theorem.

THEOREM I. *Consider a fixed pipe as described above and consider the totality of all solutions $u(x, t)$ satisfying the given boundary conditions. Then to any value of $\mu > 0$, no matter how small, there exist two finite numbers M_1^* and M_2^* which depend only on the viscosity parameter μ, on the net flux N and on the shape of the boundary Σ of the pipe (but not depending on the special solution considered) such that*

$$K(u) < M_1^* \tag{39}$$

$$\int_t^{t+1} I(u)\,dt < M_2^* \tag{40}$$

hold for every one of the solution considered, eventually, i.e. for all sufficiently large values of t.

In order to prove this theorem we first prove the following:

THEOREM II. *To any value of $\mu > 0$, no matter how small, there exist two* *finite numbers M_1 and M_2 and a field $a(x, \mu)$, where a satisfies the same conditions as u and $a = u$ on the boundary, which depend only on the viscosity μ, on the net flux N and on the shape of the boundary Σ of Ω (but not depending on the special solution considered) such that*

$$K(u-a) < M_1\,, \tag{41}$$

$$\int_t^{t+1} I(u-a)\,dt < M_2 \tag{42}$$

hold eventually for everyone of the solutions considered.

Theorem II, of course, implies the existence of eventual bounds for the quantities $K(u)$ and $\int_t^{t+1} I(u)\,dt$ which occur in Theorem I. To see this we merely have to use the triangle inequality for the Hilbert metrics $K^{1/2}$ and $I^{1/2}$. It is rather impossible to establish these latter bounds without recourse to a properly chosen auxiliary field $a(x, \mu)$.

3. *Magnetic Energy*

As already pointed out, we restrict the coefficient of magnetic diffusivity $\eta[\sim(1/\sigma)]$ to be uniform in space in this section for the purpose of discussion on magnetic energy. Multiply (6) by fB_i, sum on i; and integrate over the section Ω_1 of the pipe. Then the rate of increase of magnetic energy is found to be

$$\frac{d\tilde{W}_\alpha}{dt} = \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \epsilon_{ijk}\epsilon_{klm} f(u_l B_m)_{,j}\, B_i\,dx + \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \eta f B_{i,jj}\,dx. \tag{43}$$

But

$$\begin{aligned}
\frac{1}{4\alpha}\int_{-}^{\alpha} \eta f B_{i,jj}\,dx &= -\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \epsilon_{ijk}\eta f J_{k,j} B_i\,dx \\
&= -\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \eta[\epsilon_{ijk}(fJ_i B_k)_{,i} + \epsilon_{ijk}(fB_k)_{,j}\,J_i]\,dx \\
&= -\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \eta[\epsilon_{ijk}(fJ_j B_k)_{,i} + \epsilon_{ijk} f B_{k,j} J_i + \epsilon_{ijk} f_{,j} B_k J_i]\,dx \\
&= -\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} f(J_i J_{i/\sigma})\,dx - \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \eta\epsilon_{ijk} B_{k,j} J_i\,dx,
\end{aligned} \tag{44}$$

where J_i are the components of current density and have entered here from one of the Maxwell's curl equations. The divergence term on the right in the brackets [ ] vanishes on integration, by Greens theorem.

Similarly the first term on the right of (43) is

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}[\epsilon_{lim}\epsilon_{ijk}(fu_jB_kB_m)_{,l}+\epsilon_{ijk}\epsilon_{ilm}(fB_m)_{,l}\,u_jB_k]\,dx$$

$$=-\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\epsilon_{ijk}fJ_jB_ku_i\,dx+\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}(\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx, \tag{45}$$

the divergence term vanishing on integration, as before.

Substituting from (44) and (45) into (43) we obtain

$$\frac{d\tilde{W}_\alpha}{dt}=-\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\eta f(J_iJ_i)\,dx-\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\epsilon_{ijk}J_jB_ku_i\,dx$$
$$-\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\eta\epsilon_{ijk}f_{,j}B_kJ_i\,dx+\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}(\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx. \tag{46}$$

The symbol $\sim$ is introduced because of the presence of f in the integrand.

LEMMA 1. *For each fixed finite t,*

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx \qquad \textit{is bounded for all} \qquad \alpha>\alpha_0 \tag{47}$$

and

$$\operatorname*{limit}_{\alpha\to\infty}\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx=0. \tag{48}$$

PROOF. In view of (32) we may split the given integral into two integrals, namely,

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx=\frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx+\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx. \tag{49}$$

Notice that

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx\leqslant\frac{1}{2\alpha\Delta\alpha}\int_{\alpha-\Delta\alpha}^{\alpha}|B_k|\,|J_i|\,dx$$
$$\leqslant\frac{1}{2}\left(\frac{1}{\alpha\Delta\alpha}\int_{-\alpha}^{\alpha}B_kB_k\,dx\right)^{1/2}\left(\frac{1}{\alpha\Delta\alpha}\int_{-\alpha}^{\alpha}J_iJ_i\,dx\right)^{1/2}. \tag{50}$$

Set $\Delta\alpha=\alpha^{1/2}$. Then clearly the right-hand side is bounded for all sufficiently large values of α. A similar analysis shows that

$$\frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha}\epsilon_{ijk}f_{,j}B_kJ_i\,dx$$

is also bounded for all $\alpha > \alpha_0$. To show that

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \epsilon_{ijk} f_{,j} B_k J_i \, dx = 0,$$

we again consider (49) and observe that with $\Delta\alpha = \alpha^{1/2}$, (50) gives

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} \epsilon_{ijk} B_k f_{,j} J_i \, dx \leqslant \frac{1}{2}\left[\frac{1}{\alpha^{1/2}\alpha} 0(\alpha)\right]^{1/2} \left[\frac{1}{\alpha^{1/2}\alpha} 0(\alpha)\right]^{1/2}$$
$$\leqslant \frac{1}{2\alpha^{1/2}}\left[\frac{1}{\alpha} 0(\alpha)\right]^{1/2} \left[\frac{1}{\alpha} 0(\alpha)\right]^{1/2}.$$

Clearly for each fixed t,

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} \epsilon_{ijk} f_{,j} B_k J_i \, dx$$

approaches zero as α tends to infinity. Similarly

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} \epsilon_{ijk} f_{,j} B_k J_i \, dx = 0.$$

This proves Lemma 1.

Lemma 2. *For each fixed finite t,*

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx \quad \text{is bounded for all} \quad \alpha > \alpha_0, \tag{51}$$

and

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx = 0. \tag{52}$$

Proof. As in the preceding lemma we split the given integral into two integrals, namely,

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx = \frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx$$
$$+ \frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx. \tag{53}$$

Notice again that

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (\epsilon_{ijk} u_j B_k)(\epsilon_{ilm} f_{,l} B_m)\, dx$$
$$\leqslant \frac{1}{2\alpha\Delta\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} u_j B_k B_k \, dx$$
$$\leqslant \frac{1}{2}\left(\frac{1}{\alpha\Delta\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} B_i B_i B_k B_k \, dx\right)^{1/2} \left(\frac{1}{\alpha\Delta\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} u_j u_j \, dx\right)^{1/2}$$
$$\leqslant \frac{1}{2}\left(\frac{1}{\alpha\Delta\alpha}\int_{-\alpha}^{\alpha} B_i B_i B_k B_k \, dx\right)^{1/2} \left(\frac{1}{\alpha\Delta\alpha}\int_{-\alpha}^{\alpha} u_j u_j \, dx\right)^{1/2}. \tag{54}$$

Set $\Delta\alpha = \alpha^{1/2}$, as before. Then it is immediate that the right-hand side in (54) is bounded for all sufficiently large values of α. A similar argument will show that

$$\frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx$$

is also bounded for all $\alpha > \alpha_0$. In order to show that

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx = 0,$$

we again consider (53) and observe that with $\Delta\alpha = \alpha^{1/2}$, (54) gives

$$\begin{aligned}\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx &\leqslant \frac{1}{2}\left[\frac{1}{\alpha^{1/2}\alpha}\,0(\alpha)\right]^{1/2}\left[\frac{1}{\alpha^{1/2}\alpha}\,0(\alpha)\right]^{1/2}\\ &\leqslant \frac{1}{2\alpha^{1/2}}\left[\frac{1}{\alpha}\,0(\alpha)\right]^{1/2}\left[\frac{1}{\alpha}\,0(\alpha)\right]^{1/2}.\end{aligned}$$

Clearly for each fixed t,

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx$$

approaches zero as α tends to infinity. Similarly

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx = 0.$$

This concludes the proof of Lemma 2.

Integrating (46) with respect to t from 0 to t, we obtain

$$\begin{aligned}\tilde{W}_\alpha = &-\frac{1}{4}\int_0^t \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \eta f(J_iJ_i)\,dx\,dt - \frac{1}{4}\int_0^t \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \epsilon_{ijk}J_jB_ku_i\,dx\,dt\\ &-\frac{1}{4}\int_0^t \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \eta\epsilon_{ijk}f_{,j}B_kJ_i\,dx\,dt\\ &+\frac{1}{4}\int_0^t \frac{1}{\alpha}\int_{-\alpha}^{\alpha} (\epsilon_{ijk}u_jB_k)(\epsilon_{ilm}f_{,l}B_m)\,dx\,dt + \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} B_{0i}B_{0i}\,dx, \quad (55)\end{aligned}$$

where $B_{0i} = B(x, y, z, 0)$. Taking the limit as α approaches infinity and using the "bounded convergence" theorem, we obtain

$$W = W_0 - \int_0^t U\,dt - \frac{1}{4}\operatorname*{limit}_{\alpha\to\infty}\int_0^t \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \epsilon_{ijk}J_jB_ku_i\,dx\,dt. \qquad (56)$$

Clearly the existence of every other limit in (56) implies the existence of

$$\operatorname*{limit}_{\alpha\to\infty} \int_0^t \frac{1}{\alpha} \int_{-\alpha}^{\alpha} \epsilon_{ijk} J_j B_k u_i \, dx \, dt.$$

$W_0 = \text{limit}_{\alpha\to\infty} 1/4\alpha \int_{-\alpha}^{\alpha} B_{0i} B_{0i} \, dx$ is the initial energy of the magnetic field.

(56) is the integral form of the energy equation for the magnetic field. The last term on the right expresses the work done by the material against the ponderomotive force during the motion in time t.

As a remark we may mention that the temperature distribution in the fluid may be altered by the presence of the magnetic field, because the components u_i are affected, but the total energy of the conducting fluid is not. The kinetic energy removed by the force of the magnetic field is exactly equal to the heat generated by the current.

4. *Formal Solution*

We emphasize the fact that the electrical conductivity σ is now finite, positive and nonuniform. Let the boundary of the pipe belong to a sufficiently high differentiability class say C''', so as to render the following statements valid. Denote by $s(x) = s(x_2, x_3)$ the distance of a point x from the boundary Σ. There exists a number $q > 0$ such that $s(x)$ is of differentiability class C'' in $\Omega_q + \Sigma$, where

$$\Omega_q = [x : x \in \Omega, s(x) \leqslant q] \tag{57}$$

is the boundary strip in Ω of width q. The gradient of s is one,

$$s_{,i} s_{,i} = 1. \tag{58}$$

The key to the proof of this theorem is the following lemma.

LEMMA 3. *To any $\epsilon > 0$, there exists a field $a(x)$ of class C'' in $\Omega + \Sigma$ which is solenoidal in Ω, which assumes the boundary values prescribed for u on Σ and which satisfies*

$$\int_{x_1=\text{const}} a \, dS = N, \tag{59}$$

$$| a_{i,j} | \leqslant \frac{\epsilon}{s^2}. \tag{60}$$

PROOF. To prove this lemma we begin with a very simple remark. We suppose that q in (57) is chosen so small that the area of Ω_q (on the cross-section) is less than half the area A of the cross-section. The same is necessarily true of any Ω_δ, $\delta \leqslant q$. It suffices to construct an a which satisfies (25) together with an additional property:

$$a_1 = 1 \quad \text{in} \quad \Omega - \Omega_\delta, \quad 0 \leqslant a_1 \leqslant 1 \quad \text{in} \quad \Omega_\delta. \tag{61}$$

Then the value of integral (59) obviously equals θA, $\frac{1}{2} < \theta < 1$. Clearly multiplication of a_1 with the factor $N/\theta A$ yields a function a_1 which satisfies all the requirements of the lemma including (59) where the ϵ of (60) is multiplied by $N/\theta A$. If we start with $\epsilon A/2N$ in place of ϵ, then (60) is leterally satisfied, since

$$\frac{N}{\theta A} \cdot \frac{\epsilon A}{2N} < \epsilon.$$

Therefore it suffices to construct a vector field a which satisfies (25) and with (59) replaced by (61). Then in order to have a_1 also satisfy (60) we employ a twice continuously differentiable function $\Phi(r)$, $0 \leqslant r \leqslant 1$ and $0 \leqslant \Phi(r) \leqslant 1$ and we let

$$a_1(x_2, x_3) = \Phi\left(\frac{s}{\delta}\right), \qquad s = s(x_2, x_3). \tag{62}$$

Then Φ satisfies the following properties:

$$\Phi(0) = 0, \tag{63}$$

$$\Phi(1) = 1, \tag{64}$$

$$\Phi_{,i}(1) = 0, \tag{65}$$

$$\Phi_{,ii}(1) = 1. \tag{66}$$

Set $(s/\delta) = \tau$, $0 < \tau \leqslant 1$. Then on differentiation, (62) yields

$$a_{1,i} = \Phi_{,i} \cdot s_{,i} \frac{1}{\delta}. \tag{67}$$

In order to have $| a_{1,i} | < (\epsilon/s^2)$, it suffices to have $\Phi(\tau)$ satisfy

$$\Phi_{,i}(\tau) \leqslant \frac{2}{\tau^2}, \qquad 0 < \tau \leqslant 1, \tag{68}$$

and we choose $\delta = \min(\epsilon/2, q)$. In fact from (67) and (58) we have

$$| a_{1,i} | \leqslant \Phi_{,i} \frac{| s_{,i} |}{\delta} \leqslant \frac{2}{\tau^2 \delta} \leqslant \frac{2\delta}{s^2} \leqslant \frac{\epsilon}{s^2}. \tag{69}$$

The existence of such a function Φ satisfying (63) through (66) and (69) such that $0 \leqslant \Phi \leqslant 1$ can be easily shown and the proof may therefore be omitted. This proves the lemma.

From the well-known inequality

$$\int_0^q \left[\frac{f(s)}{s}\right]^2 ds \leqslant 4 \int_0^q f'^2(s)\, ds,$$

$f \in C'$ in $0 \leqslant s \leqslant q$, $f(0) = 0$, it now easily follows that there exists a finite constant D such that

$$\int_{\Omega\delta} \left[\frac{f(x)}{s(x)}\right]^2 dx_2\, dx_3 \leqslant D \int_{\Omega\delta} f_{,i} f_{,i}\, dx_2\, dx_3 . \tag{70}$$

Define $v = u - a$, where a is the field constructed above for a given $\epsilon > 0$. Then v satisfies (8) through (10) together with

$$\int_{x_1=\text{const}} v_1\, dS = 0. \tag{71}$$

The momentum transport equations expressed in terms of the difference vector field are

$$\frac{\partial v_i}{\partial t} + v_{i,j}v_j + a_{i,j}v_j + v_{i,1}a_1$$
$$= -p_{,i} - \sigma B_j B_j v_i - \sigma B_j B_j a_i + \mu v_{i,jj} + \mu a_{i,jj} . \tag{72}$$

Multiply both sides of (72) by $v_i f$, sum on i, and integrate over the section Ω_1 of the pipe. Then a straight-forward, though lengthy, calculation gives

$$\begin{aligned}\frac{d\tilde{K}_\alpha(v)}{dt} &= \tilde{Q}_\alpha(v) - \mu\tilde{I}_\alpha(v) + \tilde{L}_\alpha(v) + \tilde{H}_\alpha(v) + \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_1 f_{,1} p\, dx \\ &\quad + \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} v_1 f_{,1} v_i v_i\, dx + \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_1 v_1 f_{,1} a_1\, dx \\ &\quad + \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} a_1 f_{,1} v_i v_i\, dx - \frac{\mu}{2\alpha}\int_{-\alpha}^{\alpha} v_i f_{,1} v_{i,1}\, dx, \end{aligned} \tag{73}$$

where $\tilde{Q}_\alpha$ is the quadratic form

$$\tilde{Q}_\alpha(v) = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_{1,j} v_j f a_1\, dx, \tag{74}$$

$\tilde{H}_\alpha$ is the quadratic form

$$\tilde{H}_\alpha(v) = -\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} \sigma f B_j B_j v_i v_i\, dx \tag{75}$$

and $\tilde{L}_\alpha$ is the linear form

$$\tilde{L}_\alpha = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} f(\mu a_{1,jj} v_1 - 2\sigma B_j B_j a_i a_i)\, dx. \tag{76}$$

The symbol $\sim$ is introduced because of the presence of f in the integrand. It will now be shown that for each fixed finite t, the integrals still appearing on the right-hand side of (73) are bounded for all sufficiently large α and that these integrals tend to zero in the limit as α goes to infinity.

LEMMA 4. *For each fixed finite t,*

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx \qquad \textit{is bounded for all} \qquad \alpha > \alpha_0\,, \tag{77}$$

and

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx = 0. \tag{78}$$

PROOF. In view of (32) we may split the given integral into two integrals, namely

$$\frac{1}{4\alpha}\int_{-\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx = \frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} v_i v_i v_1 f_{,1}\,dx + \frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx. \tag{79}$$

By the Schwarz inequality and (37) one obtains

$$\begin{aligned}\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx &\leqslant \frac{1}{4}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha}(v_i v_i v_j v_j f_{,1}^2)\,dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_1 v_1\,dx\right]^{1/2}\\ &\leqslant \frac{1}{4}\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v^4\psi^4\,dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v_i v_i\,dx\right]^{1/2}.\end{aligned} \tag{80}$$

By the Sobolev inequality [12] applied to the section Ω_1,

$$\begin{aligned}\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v^4\psi^4\,dx\right]^{1/2} &\leqslant A\left[\frac{1}{\alpha^{1/2}}\int_{-\alpha}^{\alpha} v_j v_j \psi^2\,dx\right]^{1/4}\left[\frac{1}{\alpha^{1/2}}\int_{-\alpha}^{\alpha}\{(v_i\psi)_{,k}\}^2\,dx\right]^{3/4}\\ &\leqslant 2A\left[\frac{1}{\alpha^{1/2}}\int_{-\alpha}^{\alpha} v_j v_j \psi^2\,dx\right]^{1/4}\\ &\qquad\times\left[\frac{1}{\alpha^{1/2}}\int_{-\alpha}^{\alpha}(v_{i,k}v_{i,k}\psi^2 + v_i v_i \psi_{,1}\psi_{,1})\,dx\right]^{3/4}.\end{aligned}$$

Using (37) and (38) and setting $\Delta\alpha = \alpha^{1/2}$, we obtain

$$\begin{aligned}\left[\frac{1}{\alpha}\int v^4\psi^4\,dx\right]^{1/2} &\leqslant 2A\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v_j v_j \mid \varphi_{,1}\mid dx\right]^{1/4}\\ &\times\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v_{i,k}v_{i,k}\mid\varphi_{,1}\mid dx + \frac{1}{\alpha^2}\int_{-\alpha}^{\alpha} v_i v_i\{(\mid\varphi_{,1}\mid^{1/2})_{,1}\}^2\,dx\right]^{3/4}.\end{aligned} \tag{81}$$

Since φ, $\varphi_{,1}$ and $\varphi_{,11} \mid \varphi_{,1} \mid^{-1/2}$ are all bounded, it follows from (81), (80), (30), and (29) that for each fixed finite t,

$$\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v^4\psi^4\,dx \qquad \text{and consequently} \qquad \frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx$$

is bounded for all sufficiently large α. A similar analysis would show that

$$\frac{1}{\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} v_i v_i v_1 f_{,1}\,dx$$

is also bounded for all sufficiently large α. In order to show that

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx = 0,$$

we again consider (79) and simply observe that

$$\left[\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx\right] \leqslant \frac{1}{4}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v^4\psi^4\,dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i\,dx\right]^{1/2}. \tag{82}$$

By the Sobolev inequality together with (37-38) and $\Delta\alpha = \alpha^{1/2}$, we obtain

$$\left[\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx\right] \leqslant \frac{1}{2} A\left[\frac{1}{\alpha}\,o(\alpha)\right]^{1/4}\left[\frac{1}{\alpha}\,o(\alpha) + \frac{1}{\alpha}\,o(\alpha)\right]^{3/4}\left[\frac{1}{\alpha}\,o(\alpha)\right]^{1/2}.$$

Obviously for each fixed finite t,

$$\frac{1}{4\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i v_1 f_{,1}\,dx$$

approaches zero as α tends to infinity. Similarly

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{4\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} v_i v_i v_1 f_{,1}\,dx = 0.$$

This proves Lemma 4.

LEMMA 5. *For each fixed finite t,*

$$\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_1 p f_{,1}\,dx \qquad \textit{is bounded for all} \qquad \alpha > \alpha_0 \tag{83}$$

and

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_1 p f_{,1}\,dx = 0. \tag{84}$$

PROOF. As in the preceding lemma, we split the given integral into two integrals and use (71) to obtain

$$\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_1 p f_{,1}\, dx = \frac{1}{2\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx + \frac{1}{2\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx. \tag{85}$$

But

$$\begin{aligned}\left[\frac{1}{2\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx\right] &\leqslant \left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_1 v_1\, dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})^2\, dx\right]^{1/2}\\ &\leqslant \left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} v_i v_i\, dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} (p-\bar{p})^2\, dx\right]^{1/2}\\ &\leqslant 2(\bar{C}_1 C_3)^{1/2}.\end{aligned}$$

Similarly

$$\left[\frac{1}{2\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx\right] \leqslant 2(\bar{C}_1 C_3)^{1/2}.$$

Hence $1/2\alpha \int_{-\alpha}^{\alpha} p v_1 f_{,1}\, dx$ is bounded for all $\alpha > \alpha_0$. In order to show that

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} p v_1 f_{,1}\, dx = 0,$$

we observe from (85) that

$$\begin{aligned}\left[\frac{1}{2\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx\right] &\leqslant \left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i\, dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})^2\, dx\right]^{1/2}\\ &\leqslant \left[\frac{1}{\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} v_i v_i\, dx\right]^{1/2}\left[\frac{1}{\alpha}\int_{-\alpha}^{\alpha} (p-\bar{p})^2\, dx\right]^{1/2}.\end{aligned}$$

Clearly

$$\left[\frac{1}{2\alpha}\int_{\alpha-\Delta\alpha}^{\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx\right]$$

approaches zero as α tends to infinity. Similarly

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{-\alpha+\Delta\alpha} (p-\bar{p})\, v_1 f_{,1}\, dx = 0.$$

Hence

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} p v_1 f_{,1}\, dx = 0.$$

This concludes the proof of Lemma 5.

By a similar and much simpler analysis, it is easily shown that for each fixed finite t, all the remaining integrals are bounded and they all tend to zero in the limit.

Thus (73) may be written as

$$\frac{d\tilde{K}_\alpha}{dt} = \tilde{Q}_\alpha(v) - \mu\tilde{I}_\alpha(v) + \tilde{L}_\alpha(v) + \tilde{H}_\alpha(v) + \tilde{\epsilon}_\alpha . \tag{86}$$

where $\tilde{\epsilon}_\alpha$ is bounded for each fixed t and $\operatorname{limit}_{\alpha\to\infty} \tilde{\epsilon}_\alpha = 0$. Consider

$$\tilde{Q}_\alpha(v) = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} v_{1,j} v_j f a_1 \, dx.$$

By virtue of Gauss-Green theorem and the inequality

$$\sum_i \sum_j | v_i | \, | v_j | \leqslant 3 \sum_i v_i v_i , \tag{87}$$

one easily obtains

$$\begin{aligned}\tilde{Q}_\alpha(v) &\leqslant \frac{3}{2\alpha}\int_{-\alpha}^{\alpha} f v_i v_i \, | a_{1,j} | \, dx \\ &\leqslant \frac{3}{2\alpha}\int_{-\alpha}^{\alpha} f \left\{ \iint_{\Omega\delta} v_i v_i \, | a_{1,j} | \, dx_2 \, dx_3 \right\} dx_1 . \end{aligned} \tag{88}$$

Lemma 3 and subsequently (70) are now applicable to (88) giving us

$$\begin{aligned}\tilde{Q}_\alpha(v) &\leqslant \frac{3}{2\alpha}\int_{-\alpha}^{\alpha} f \left\{ \iint_{\Omega\delta} \frac{v_i v_i}{s^2} \, dx_2 \, dx_3 \right\} dx_1 \\ &\leqslant \frac{3\epsilon D}{2\alpha}\int_{-\alpha}^{\alpha} f v_{i,j} v_{i,j} \, dx \\ &\leqslant 3\epsilon D \tilde{I}_\alpha . \end{aligned} \tag{89}$$

Choose $\epsilon = (\mu/6D)$, then (89) reduces to

$$\tilde{Q}_\alpha(v) \leqslant \frac{\mu}{2} \tilde{I}_\alpha . \tag{90}$$

We next consider

$$\tilde{L}_\alpha(v) = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} f(\mu a_{1,jj} v_1 - 2\sigma B_j B_j a_i a_i) \, dx.$$

Since $1/\alpha \int_{-\alpha}^{\alpha} \sigma B_j B_j a_i a_i \, dx$ is positive definite and

$$\operatorname*{limit}_{\alpha\to\infty} \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \sigma f B_j B_j a_i a_i \, dx \text{ exists,}$$

we infer that

$$\tilde{L}_\alpha(v) \leqslant \frac{\mu}{2\alpha} \int_{-\alpha}^{\alpha} a_{1,jj} v_1 f \, dx. \tag{91}$$

By the Schwarz inequality, (91) reduces to

$$\tilde{L}_\alpha(v) \leqslant m(\tilde{K}_\alpha)^{1/2}, \tag{92}$$

where m depends only on the shape of the boundary and on μ. Since $\tilde{H}_\alpha$ is a negative definite quadratic form for all admissible B_i and v_i and $\text{limit}_{\alpha\to\infty} \tilde{H}_\alpha$ exists, (86) implies

$$\frac{d\tilde{K}_\alpha}{dt} \leqslant m(\tilde{K}_\alpha)^{1/2} - \frac{\mu}{2} \tilde{I}_\alpha + \tilde{\epsilon}_\alpha . \tag{93}$$

By the Poincare's inequality [3] applied to Ω_1, (93) reduces to

$$\frac{d\tilde{K}_\alpha}{dt} \leqslant m(\tilde{K}_\alpha)^{1/2} - \frac{\mu}{2C} \tilde{K}_\alpha + \tilde{\epsilon}_\alpha , \tag{94}$$

where C is a finite constant independent of the size of Ω_1. Integrating (94) with respect to t in a finite t-interval (t, t') and droping the symbol $\sim$ (this does not alter the inequality) we obtain

$$K_\alpha(t') \leqslant K_\alpha(t) + \int_t^{t'} \left[m(K_\alpha)^{1/2} - \frac{\mu}{2C} K_\alpha \right] ds + \int_t^{t'} \epsilon_\alpha(s) \, ds. \tag{95}$$

Since $K_\alpha(t)$ and $\epsilon_\alpha(t)$ are bounded for each fixed finite t, we can apply the "bounded convergence " theorem to (95) to obtain, as α goes to infinity

$$K(t') - K(t) \leqslant \int_t^{t'} \left[mK^{1/2}(s) - \frac{\mu}{2C} K(s) \right] ds, \qquad 0 \leqslant t \leqslant t'. \tag{96}$$

The integral of (96) as a function of the variable K is positive if

$$0 < K < K_0 , \qquad mK_0^{1/2} - \frac{\mu}{2C} K_0 = 0 \tag{97}$$

and decreases (and is negative) for $K > K_0$. From these facts and from (96) we can deduce the theorem. Observe that (96) implies

$$\operatorname*{limit\,inf}_{t\to\gamma-0} K(t) \geqslant K(\gamma) \tag{98}$$

for any $\gamma > 0$. Hence, if $K(s) > K_0$, $K(s) > K_0$ must hold in some interval

$\beta < s \leqslant \gamma$. In any interval $\beta < s \leqslant \gamma$ in which $K(s) > K_0$ holds, $K(s)$ decreases by virtue of (96). Furthermore, it follows from (96), $t = \beta$, $t' > \beta$, $t' \to \beta$, that

$$K(\beta) \geqslant \operatorname*{limit\,sup}_{t \to \beta + 0} K(t') \tag{99}$$

and hence, that also $K_1(\beta) > K_0$. Both these facts evidently imply that if $K(\gamma) > K_0$, then $K(s) > K_0$ must hold for all $s \leqslant \gamma$ and $K(s)$ decreases in $0 \leqslant s \leqslant \gamma$ with increasing s. From the properties of the integrand, it now follows that exactly the same conclusion holds if K_0 is replaced by any fixed number $K_1 > K_0$.

The conclusion can be formulated: If $K(t) \leqslant K_1$ holds for some t, then it must hold for all larger t. It is easily shown from (96) that to any given number $K_1 > K_0$, there is at least one moment of time, say t_1 such that $K(t_1) \leqslant K_1$. The theorem is thereby proved as far as K is concerned. The assertion concerning $\int_t^{t+1} I(s)\, ds$ now easily follows from (93).

The problem of flow through an infinitely long cylindrical pipe, the way we have considered it here (we did not require vanishing of velocity at infinity or finite kinetic and magnetic energy) is the magnetohydrodynamic analogue of the flow problem of infinite type in hydrodynamics [2]. For the corresponding flow problem of finite type (in these problems K, W, and μI are simply the total kinetic and magnetic energies and the total dissipation of kinetic energy, respectively, in the container at each moment of time) the above theorem has already been proved [4].

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